Huneke–Wiegand conjecture of rank one with the change of rings

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RIMS Workshop The 35th Symposium on Commutative Algebra

December 5, 2013

Introduction

1. R an integral domain
2. M, N finitely generated **torsionfree** R-modules

Question

When is the tensor product $M \otimes_R N$ torsionfree?

Conjecture 1.1 (Huneke–Wiegand conjecture [4])

Let R be a Gorenstein local domain. Let M be a maximal C–M R-module. If $M \otimes_R \mathrm{Hom}_R(M, R)$ is torsionfree, then M is free.

Conjecture 1.2

Let R be a Gorenstein local domain with $\dim R = 1$ and I an ideal of R. If $I \otimes_R \mathrm{Hom}_R(I, R)$ is torsionfree, then I is principal.

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In my lecture we are interested in the question of what happens if we replace $\mathrm{Hom}_R(I, R)$ by $\mathrm{Hom}_R(I, \mathrm{K}_R)$.

Conjecture 1.3

Let R be a C–M local ring with $\dim R = 1$ and assume $\exists\, \mathrm{K}_R$. Let I be a faithful ideal of R. If $I \otimes_R \mathrm{Hom}_R(I, \mathrm{K}_R)$ is torsionfree, then $I \cong R$ or K_R as an R-module.

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Theorem 1.4 (Main Theorem)

Let R be a C–M local ring with $\dim R = 1$ and assume $\exists\, \mathrm{K}_R$. Let I be a faithful ideal of R.

(1) Assume that the canonical map

$$t : I \otimes_R \mathrm{Hom}_R(I, \mathrm{K}_R) \to \mathrm{K}_R,\ x \otimes f \mapsto f(x)$$

is an isomorphism. If $r, s \geq 2$, then $\mathrm{e}(R) > (r+1)s \geq 6$, where $r = \mu_R(I)$ and $s = \mu_R(\mathrm{Hom}_R(I, \mathrm{K}_R))$.

(2) Suppose that $I \otimes_R \mathrm{Hom}_R(I, \mathrm{K}_R)$ is torsionfree. If $\mathrm{e}(R) \leq 6$, then $I \cong R$ or K_R.

Corollary 1.5

Let R be a C–M local ring with $\dim R \geq 1$. Assume that $R_\mathfrak{p}$ is Gorenstein and $\mathrm{e}(R_\mathfrak{p}) \leq 6$ for every height one prime $\mathfrak{p}$. Let I be a faithful ideal of R. If $I \otimes_R \mathrm{Hom}_R(I, R)$ is reflexive, then I is principal.

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Let R be a C–M local ring with $\dim R = 1$ and assume $\exists\, \mathrm{K}_R$. Let I be a faithful ideal of R.

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Contents

Notation

In what follows, unless other specified, we assume

1. $(R, \mathfrak{m})$ a C–M local ring, $\dim R = 1$
2. $F = \mathrm{Q}(R)$ the total ring of fractions of R.
3. $\mathcal{F} = \{I \mid I \text{ is a fractional ideal such that } FI = F\}$
4. $\exists$ a canonical module K_R of R
5. $M^\vee = \mathrm{Hom}_R(M, \mathrm{K}_R)$ for each R-module M
6. $\mu_R(M) = \ell_R(M/\mathfrak{m}M)$ for each R-module M

Change of rings

Let $I \in \mathcal{F}$. Denote by

$$t : I \otimes_R I^\vee \to \mathrm{K}_R,\ x \otimes f \mapsto f(x).$$

Then the diagram

$$\begin{array}{ccc} F \otimes_R (I \otimes_R I^\vee) & \xrightarrow{\cong} & F \otimes_R \mathrm{K}_R \\ {\scriptstyle\alpha}\uparrow & & \uparrow \\ I \otimes_R I^\vee & \xrightarrow{t} & \mathrm{K}_R \end{array}$$

is commutative. Hence

$$T := \mathrm{T}(I \otimes_R I^\vee) = \mathrm{Ker}\, t.$$

Lemma 2.1

$I \otimes_R I^\vee$ is torsionfree $\Longleftrightarrow$ $t : I \otimes_R I^\vee \longrightarrow \mathrm{K}_R$ is injective.

We set $L = \mathrm{Im}(I \otimes_R I^\vee \xrightarrow{t} \mathrm{K}_R)$.
Consider

$$0 \to T \to I \otimes_R I^\vee \xrightarrow{t} L \to 0.$$

Hence

$$L^\vee \cong (I \otimes_R I^\vee)^\vee = \mathrm{Hom}_R(I, I^{\vee\vee}) \cong I : I =: B \subseteq F.$$

Lemma 2.1

$I \otimes_R I^\vee$ is torsionfree $\Longleftrightarrow t : I \otimes_R I^\vee \longrightarrow \mathrm{K}_R$ is injective.

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Let $R \subseteq S \subseteq B$. Then I is also a fractional ideal of S.

$$L = L^{\vee\vee} = B^{\vee} = \mathrm{K}_B \subseteq S^{\vee} = \mathrm{K}_S \quad \text{and}$$

$$\begin{aligned} \mathrm{Hom}_S(I, \mathrm{K}_S) &= \mathrm{Hom}_S(I, \mathrm{Hom}_R(S, \mathrm{K}_R)) \\ &\cong \mathrm{Hom}_R(I \otimes_S S, \mathrm{K}_R) = \mathrm{Hom}_R(I, \mathrm{K}_R) = I^{\vee}. \end{aligned}$$

$$\begin{array}{ccc} I \otimes_S \mathrm{Hom}_S(I, \mathrm{K}_S) & \xrightarrow{t_S} & \mathrm{K}_S \\ \rho \uparrow & & \iota \uparrow \\ I \otimes_R I^\vee & \xrightarrow{t} & L \end{array}$$

where $\rho(x \otimes f) = x \otimes f$.

Lemma 2.2

Let $I \in \mathcal{F}$ and $R \subseteq S \subseteq B = I : I$. If $I \otimes_R I^\vee$ is torsionfree, then $I \otimes_S \mathrm{Hom}_S(I, \mathrm{K}_S)$ is a torsionfree S-module and

$$\rho : I \otimes_R I^\vee \to I \otimes_S \mathrm{Hom}_S(I, \mathrm{K}_S)$$

is bijective.
In particular, if $S = B$, then

$$t_B : I \otimes_B \mathrm{Hom}_B(I, \mathrm{K}_B) \to \mathrm{K}_B, \quad x \otimes f \mapsto f(x)$$

is an isomorphism of B-modules.

Proposition 2.3 (**Change of rings**)

Let $I \in \mathcal{F}$ and assume that $I \otimes_R I^\vee$ is torsionfree. If there exists $R \subseteq S \subseteq B$ such that $I \cong S$ or K_S as an S-module, then $I \cong R$ or K_R as an R-module.

Proof.

Suppose $I \cong S$ and consider

$$I \otimes_R I^\vee \overset{\rho}{\cong} I \otimes_S \mathrm{Hom}_S(I, \mathrm{K}_S) \cong \mathrm{Hom}_S(I, \mathrm{K}_S) \cong I^\vee.$$

Then $\mu_R(I) \cdot \mu_R(I^\vee) = \mu_R(I^\vee)$, so that $I \cong R$, since $\mu_R(I) = 1$. □

Proof of Theorem 1.4

Theorem 1.4 (Main Theorem)

Let R be a C–M local ring with $\dim R = 1$ and assume $\exists\, \mathrm{K}_R$. Let I be a faithful ideal of R.

(1) Assume that $I \otimes_R I^\vee \cong \mathrm{K}_R$.
If $r, s \geq 2$, then $e > (r+1)s \geq 6$,
where $e = \mathrm{e}(R)$, $r = \mu_R(I)$, and $s = \mu_R(I^\vee)$.

(2) Suppose that $I \otimes_R I^\vee$ is torsionfree.
If $\mathrm{e}(R) \leq 6$, then $I \cong R$ or K_R.

Proof of assertion (1) of Theorem 1.4

Choose $f \in \mathfrak{m}$ such that fR is a reduction of $\mathfrak{m}$. Let

$$S = R/fR, \quad \mathfrak{n} = \mathfrak{m}/fR \quad \text{and} \quad M = I/fI.$$

Hence

$$\mu_S(M) = r, \quad \mathrm{r}_S(M) = \ell_S((0) :_M \mathfrak{n}) = s.$$

We write $M = Sx_1 + Sx_2 + \cdots + Sx_r$ and look at

$$(\sharp_0) \qquad 0 \to X \to S^{\oplus r} \xrightarrow{\varphi} M \to 0, \qquad \varphi(\mathbf{e_i}) = x_i.$$

We get

$$(\sharp_1) \qquad 0 \to M^\vee \to \mathrm{K}_S^{\oplus r} \to X^\vee \to 0,$$

$$(\sharp_2) \qquad 0 \to \mathrm{Hom}_S(M, M) \to M^{\oplus r} \to \mathrm{Hom}_S(X, M).$$

Proof of assertion (1) of Theorem 1.4

Because $S = \mathrm{Hom}_S(M, M)$, we have by $(\sharp_2)$

$$(\sharp_3) \quad 0 \to S \xrightarrow{\psi} M^{\oplus r} \to \mathrm{Hom}_S(X, M),$$

where $\psi(1) = (x_1, x_2, \ldots, x_r)$.
By

$$(\sharp_0) \quad 0 \to X \to S^{\oplus r} \xrightarrow{\varphi} M \to 0.$$

we get

$$\ell_S(X) = r{\cdot}\ell_S(S) - \ell_S(M) = re - e = (r-1)e.$$

Proof of assertion (1) of Theorem 1.4

By

$$(\sharp_1) \qquad 0 \to M^\vee \to \mathrm{K}_S^{\oplus r} \to X^\vee \to 0,$$

we have

$$q := \mu_S(X^\vee) \geq \mu_S(\mathrm{K}_S^{\oplus r}) - \mu_S(M^\vee) = r \cdot \mu_S(\mathrm{K}_S) - \mathrm{r}_S(M).$$

Therefore

$$(r-1)e = \ell_S(X) \geq \ell_S((0) :_X \mathfrak{n}) = q \geq r^2 s - s = s(r^2 - 1).$$

Thus

$$e \geq s(r+1),$$

since $r \geq 2$.

Proof of assertion (1) of Theorem 1.4.

Now suppose $e = s(r+1)$. Then $\mathfrak{n}\cdot \mathrm{Hom}_S(X, M) = (0)$. By

$$(\sharp_3) \qquad 0 \to S \xrightarrow{\psi} M^{\oplus r} \to \mathrm{Hom}_S(X, M),$$

we have

$$\mathfrak{n}\cdot M^{\oplus r} \subseteq S\cdot(x_1, x_2, \dots, x_r).$$

Hence $\mathfrak{n}M \subseteq \mathfrak{a}_i M$, where $\mathfrak{a}_i = (0) : (x_j \mid 1 \le j \le r,\ j \ne i) \subseteq \mathfrak{n}$. Therefore

$$\mathfrak{n}M = \mathfrak{a}_i M \quad \text{for} \quad 1 \le \forall i \le r,$$

so that $\mathfrak{n}^2 M = (\mathfrak{a}_1\mathfrak{a}_2)M = (0)$. Thus $\mathfrak{n}M \subseteq (0) :_M \mathfrak{n}$. Consequently

$$\begin{aligned} s = \mathrm{r}_S(M) = \ell_S((0) :_M \mathfrak{n}) \ &\ge\ \ell_S(\mathfrak{n}M) = \ell_S(M) - \ell_S(M/\mathfrak{n}M) \\ &=\ e - r = s(r+1) - r. \end{aligned}$$

Hence $0 \ge rs - r = r(s-1)$, which is impossible. □

Corollary 3.1

Let R be a Gorenstein local ring with $\dim R = 1$ and $\mathrm{e}(R) \leq 6$. Let I be a faithful ideal of R. If $I \otimes_R \mathrm{Hom}_R(I, R)$ is torsionfree, then I is principal.

Theorem 3.2

Let $(R, \mathfrak{m})$ be a C–M local ring with $\dim R = 1$ and assume that $\mathfrak{m}\overline{R} \subseteq R$. Let I be a faithful fractional ideal of R. If $I \otimes_R I^\vee$ is torsionfree, then $I \cong R$ or K_R.

Theorem 3.3

Let R be a C–M local ring with $\dim R = 1$. Assume $\exists\, \mathrm{K}_R$ and $\mathrm{v}(R) = \mathrm{e}(R)$. Let I be a faithful ideal of R. If $I \otimes_R I^\vee \cong \mathrm{K}_R$, then $I \cong R$ or K_R.

Let k be a field.

Proposition 3.4

Let $R = k[[t^a, t^{a+1}, \ldots, t^{2a-1}]]$ $(a \geq 1)$ be the semigroup ring and let $I \neq (0)$ be an ideal of R. If $I \otimes_R I^\vee$ is torsionfree, then $I \cong R$ or K_R.

Corollary 3.5

Let $R = k[[t^a, t^{a+1}, \ldots, t^{2a-2}]]$ $(a \geq 3)$ be the semigroup ring and let I be an ideal of R. If $I \otimes_R \mathrm{Hom}_R(I, R)$ is torsionfree, then I is principal.

Proof of Corollary 3.5.

Notice that R is a Gorenstein local ring with $R : \mathfrak{m} = R + kt^{2a-1}$. Suppose that $I \not\cong R$. Then $R \subsetneq B := I : I$ and therefore $t^{2a-1} \in B$, whence

$$R \subseteq S := k[[t^a, t^{a+1}, \dots, t^{2a-1}]] \subseteq B.$$

Thanks to Lemma 2.2, $I \otimes_S \mathrm{Hom}_S(I, \mathrm{K}_S)$ is S-torsionfree, so that $I \cong S$ or $I \cong \mathrm{K}_S$ as an S-module by Proposition 3.4. Hence $I \cong R$ by Proposition 2.3, which is impossible. □

Remark 3.6

Corollary 3.5 gives a new class of one-dimensional Gorenstein local domains for which Conjecture 1.2 holds true.
For example, take $a = 5$. Then $R = k[[t^5, t^6, t^7, t^8]]$ is not a complete intersection.

Numerical semigroup rings

Setting 4.1

Let $0 < a_1 < a_2 < \cdots < a_\ell \in \mathbb{Z}$ such that $\gcd(a_1, a_2, \ldots, a_\ell) = 1$.
We set $H = \langle a_1, a_2, \ldots, a_\ell \rangle = \{\sum_{i=1}^{\ell} c_i a_i \mid 0 \leq c_i \in \mathbb{Z}\}$ and

$$R = k[[t^{a_1}, t^{a_2}, \ldots, t^{a_\ell}]] \quad \subseteq \quad V = k[[t]].$$

Let $\mathfrak{m} = (t^{a_1}, t^{a_2}, \ldots, t^{a_\ell})$ be the maximal ideal of R. We set $\mathfrak{c} = R : V$ and $c = \mathrm{c}(H)$, the conductor of H, whence $\mathfrak{c} = t^c V$. Let $a = c - 1$. Notice that $\mathrm{e}(R) = a_1 = \mu_R(V)$.

Definition 4.2

Let $I \in \mathcal{F}$. Then I is said to be a monomial ideal, if $I = \sum_{n \in \Lambda} Rt^n$ for some $\Lambda \subseteq \mathbb{Z}$.

Set

$$\mathcal{M} = \{I \in \mathcal{F} \mid I \text{ is a monomial ideal}\}.$$

Passing to the monomial ideal $t^{-q}I$ for some $q \in \mathbb{Z}$, we may assume

$$R \subseteq I \subseteq V.$$

We assume that $e = a_1 \geq 2$. Set

$$\alpha_i = \max\{n \in \mathbb{Z} \setminus H \mid n \equiv i \mod e\} \qquad (0 \leq i \leq e-1)$$

and

$$\mathcal{S} = \{\alpha_i \mid 1 \leq i \leq e-1\}.$$

Hence $\alpha_0 = -e$, $\sharp\mathcal{S} = e - 1$, $a = \max\mathcal{S}$, and $\alpha_i \geq i$ for $1 \leq i \leq e-1$.

Theorem 4.3

Let $b = \min \mathcal{S}$ and suppose $t^b \in R : \mathfrak{m}$. Let $I \in \mathcal{M}$ such that $R \subseteq I \subseteq V$. If $I \otimes_R I^\vee \cong \mathrm{K}_R$, then $I \cong R$ or K_R.

The case where $\mathrm{e}(R) = 7$

Let $I \in \mathcal{M}$ such that $R \subseteq I \subseteq V$ and set $J = \mathrm{K}_R : I$. Suppose that $\mu_R(I) = \mu_R(J) = 2$ and write

$$I = (1, t^{c_1}), \quad J = (1, t^{c_2}),$$

where $c_1, c_2 > 0$. Assume $IJ = \mathrm{K}_R$ and $\mu_R(\mathrm{K}_R) = 4$.

Theorem 5.1

$e = a_1 \geq 8.$

The case where $\mathrm{e}(R) = 7$

Let $I \in \mathcal{M}$ such that $R \subseteq I \subseteq V$ and set $J = \mathrm{K}_R : I$. Suppose that $\mu_R(I) = \mu_R(J) = 2$ and write

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Theorem 5.1

$e = a_1 \geq 8$.

Theorem 5.2

Let $R = k[[t^{a_1}, t^{a_2}, \cdots, t^{a_\ell}]]$ be a semigroup ring and suppose that $e = a_1 \leq 7$. Let $I \in \mathcal{M}$. If $I \otimes_R I^\vee$ is torsionfree, then $I \cong R$ or K_R.

Corollary 5.3

Let R be a Gorenstein numerical semigroup ring with $\mathrm{e}(R) \leq 7$ and let $I \in \mathcal{M}$. If $I \otimes_R \mathrm{Hom}_R(I, R)$ is torsionfree, then I is principal.

Theorem 5.2

Let $R = k[[t^{a_1}, t^{a_2}, \cdots, t^{a_\ell}]]$ be a semigroup ring and suppose that $e = a_1 \leq 7$. Let $I \in \mathcal{M}$. If $I \otimes_R I^\vee$ is torsionfree, then $I \cong R$ or K_R.

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Let R be a Gorenstein numerical semigroup ring with $\mathrm{e}(R) \leq 7$ and let $I \in \mathcal{M}$. If $I \otimes_R \mathrm{Hom}_R(I, R)$ is torsionfree, then I is principal.

Examples

Condition: $IJ = \mathrm{K}_R$ and $\mu_R(\mathrm{K}_R) = 4$

Example 6.1

Let $R = k[[t^8, t^{11}, t^{14}, t^{15}]]$. Then $\mathrm{K}_R = (1, t, t^3, t^4)$. We take $I = (1, t)$ and set $J = \mathrm{K}_R : I$. Then $J = (1, t^3)$, $IJ = \mathrm{K}_R$, $\mu_R(\mathrm{K}_R) = 4$, but

$$\mathrm{T}(I \otimes_R J) = R(t \otimes t^{16} - 1 \otimes t^{17}) \cong R/\mathfrak{m}.$$

Remark 6.2

In the ring R of Example 6.1 $\nexists$ monomial ideals I such that $I \ncong R, I \ncong \mathrm{K}_R$, and $I \otimes_R I^\vee$ is torsionfree.

The following ideals also satisfy

$$IJ = \mathrm{K}_R \text{ and } \mu_R(\mathrm{K}_R) = 4$$

but $I \otimes_R I^\vee$ is not torsionfree.

(1) $H = \langle 8, 9, 10, 13 \rangle, \mathrm{K}_R = (1, t, t^3, t^4), I = (1, t)$.
(2) $H = \langle 8, 11, 12, 13 \rangle, \mathrm{K}_R = (1, t, t^3, t^4), I = (1, t)$.
(3) $H = \langle 8, 11, 14, 23 \rangle, \mathrm{K}_R = (1, t^3, t^9, t^{12}), I = (1, t^3)$.
(4) $H = \langle 8, 13, 17, 18 \rangle, \mathrm{K}_R = (1, t, t^5, t^6), I = (1, t)$.
(5) $H = \langle 8, 13, 18, 25 \rangle, \mathrm{K}_R = (1, t^5, t^7, t^{12}), I = (1, t^5)$.

If $\mathrm{e}(R) \geq 9$, then Conjecture 1.3 is not true in general.

Example 6.3

Let $R = k[[t^9, t^{10}, t^{11}, t^{12}, t^{15}]]$. Then $\mathrm{K}_R = (1, t, t^3, t^4)$. Let $I = (1, t)$ and put $J = \mathrm{K}_R : I$. Then

$$J = (1, t^3), \quad \mu_R(I) = \mu_R(J) = 2, \text{ and } \mu_R(\mathrm{K}_R) = 4,$$

but $I \otimes_R I^\vee$ is torsionfree.

Thank you very much for your attention!

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